On the number of empty boxes in the Bernoulli sieve II

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Abstract

The Bernoulli sieve is the infinite "balls-in-boxes" occupancy scheme with random frequencies $P_k = W_1 \cdots W_{k-1}(1-W_k)$, where $(W_k)_{k\in\mathbb{N}}$ are independent copies of a random variable W taking values in $(0,1)$. Assuming that the number of balls equals n, let L_n denote the number of empty boxes within the occupancy range. In the paper we investigate convergence in distribution of L_n in the two cases which remained open after the previous studies. In particular, provided that $\mathbb{E}|\log W| = \mathbb{E}|\log(1-W)| = \infty$ and that the law of W assigns comparable masses to the neighborhoods of 0 and 1, it is shown that L_n weakly converges to a geometric law. This result is derived as a corollary to a more general assertion concerning the number of zero decrements of nonincreasing Markov chains. In the case that $\mathbb{E}|\log W| < \infty$ and $\mathbb{E}|\log(1-W)| = \infty$ we derive several further possible modes of convergence in distribution of L_n. It turns out that the class of possible limiting laws for L_n, properly normalized and centered, includes normal laws and spectrally negative stable laws with finite mean. While investigating the second problem we develop some general results concerning the weak convergence of renewal shot-noise processes. This allows us to answer a question asked in [19].

Keywords: Bernoulli sieve, continuous mapping theorem, convergence in distribution, depoissonization, infinite occupancy scheme, renewal shot-noise process

1 Introduction

Let $(T_k)_{k\in\mathbb{N}_0}$ be a multiplicative random walk defined by

$$T_0 := 1, \quad T_k := \prod_{i=1}^{k} W_i, \quad k \in \mathbb{N},$$

where $(W_k)_{k\in\mathbb{N}}$ are independent copies of a random variable W taking values in $(0,1)$. Let $(U_k)_{k\in\mathbb{N}}$ be independent random variables with the uniform $[0,1]$ law which are independent of the multiplicative random walk. The *Bernoulli sieve* is a random occupancy scheme in which 'balls' U_k's are allocated over infinitely many 'boxes' $(T_k, T_{k-1}]$, $k \in \mathbb{N}$. The scheme was introduced in [8]. Further investigations were made in [10, 11, 12, 13, 14, 17]. Since a particular ball falls in box $(T_k, T_{k-1}]$ with probability

$$P_k := T_{k-1} - T_k = W_1 W_2 \cdots W_{k-1}(1 - W_k),$$

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the Bernoulli sieve is also the classical infinite occupancy scheme [9, 18] with random frequencies $(P_k)_{k\in\mathbb{N}}$, where (abstract) balls are allocated over an infinite array of (abstract) boxes $1, 2, \ldots$ independently conditionally given (P_k) with probability P_j of hitting box j. Alternatively the Bernoulli sieve can be thought of as a randomized variant of the leader election procedure which appears if the law of W is degenerate at some $x \in (0,1)$ (this may be especially appropriate for the reader familiar with the analysis of algorithms).

We will use the following notation for the moments

$$\mu := \mathbb{E}|\log W| \quad \text{and} \quad \nu := \mathbb{E}|\log(1-W)|$$

which may be finite or infinite. Assuming that the number of balls equals n denote by K_n the number of occupied boxes, M_n the index of the last occupied box, and $L_n := M_n - K_n$ the number of empty boxes within the occupancy range. The present paper is a contribution towards understanding the weak convergence of L_n. With the account of the results obtained here and in some previous works on the subject we can now draw an almost complete picture (Remark 1.4 which discusses two cases where the weak convergence of L_n remains unsettled reveals what is hidden behind the word 'almost'). Depending on the behavior of the law of W near the endpoints 0 and 1 the number of empty boxes can exhibit quite a wide range of different asymptotics.

CASE $\mu < \infty$ AND $\nu < \infty$: L_n converges in distribution and in mean to some L with proper and nondegenerate law (Theorem 2.2(a) in [13] and Theorem 3.3 in [14]). Furthermore there is also convergence of all moments (Theorem 20(b) in [20]).

CASE $\mu = \infty$ AND $\nu < \infty$: L_n converges to zero in probability (Theorem 2.2(a) in [13]).

CASE $\mu < \infty$ AND $\nu = \infty$: There are several possible modes of the weak convergence of L_n, properly normalized and centered (see Theorem 1.2 of the present paper).

CASE $\mu = \infty$ AND $\nu = \infty$: The asymptotics of L_n is determined by the behavior of the ratio $\mathbb{P}\{W \le x\}/\mathbb{P}\{1-W \le x\}$, as $x \downarrow 0$. When the law of W assigns much more mass to the neighborhood of 1 than to that of 0 equivalently the ratio goes to 0, L_n becomes asymptotically large. In this situation the weak convergence result for L_n, properly normalized without centering, was obtained in [17] under a condition of regular variation. If the roles of 0 and 1 are interchanged L_n converges to zero in probability (this follows from Theorem 7.1(i) in [12] and Markov inequality). When the tails are comparable L_n weakly converges to a geometric distribution (see Theorem 1.1 of the present paper).

Also it was known that whenever $L_n \xrightarrow{d} L$, where L is a random variable with a proper and nondegenerate probability law, the law of L is mixed Poisson (Proposition 1.2 in [17]), and that L_n has the geometric distribution with parameter $1/2$ when $W \stackrel{d}{=} 1-W$ (Proposition 7.1 in [12]).

Throughout the paper geom(a) denotes a random variable which has the geometric distribution (starting at zero) with success probability a, i.e.,

$$\mathbb{P}\{\text{geom}(a) = m\} = a(1-a)^m, \quad m \in \mathbb{N}_0,$$

and $\mathcal{N}(0,1)$ denotes a random variable which has the standard normal distribution.

We are ready to state our first result which treats the case of 'comparable tails' when $\mu = \nu = \infty$.

Theorem 1.1. *Suppose $\mu = \infty$ and*

$$\lim_{n\to\infty} \frac{\mathbb{E}W^n}{\mathbb{E}(1-W)^n} = c \in (0,\infty). \tag{1}$$

Then

$$L_n \stackrel{d}{\to} L \stackrel{d}{=} \text{geom}((c+1)^{-1}), \quad n \to \infty. \tag{2}$$

In particular, relation (2) *holds whenever the tails are comparable, i.e.,*

$$\lim_{x \downarrow 0} \frac{\mathbb{P}\{1-W \le x\}}{\mathbb{P}\{W \le x\}} = c. \tag{3}$$

The situation when $\mu < \infty$ and $\nu = \infty$ is covered by Theorem 1.2 which is our second result.

Theorem 1.2. *Suppose $\nu = \infty$, and the law of $|\log W|$ is non-lattice. Set*

$$b_n := \frac{1}{\mu} \int_{[1,n]} \frac{\psi(z)}{z} \mathrm{d}z,$$

where $\psi(s) := \mathbb{E}e^{-s(1-W)}$, $s \ge 0$.
(a) *If $\sigma^2 = \mathrm{Var}\,(\log W) < \infty$ then, with $a_n := \sqrt{b_n}$, the limiting distribution of $\frac{L_n - b_n}{a_n}$ is standard normal.*
(b) *Assume that $\sigma^2 = \infty$ and*

$$\int_{[0,x]} y^2 \mathbb{P}\{|\log W| \in \mathrm{d}y\} \ \sim\ \widetilde{\ell}(x), \quad x \to \infty, \tag{4}$$

for some $\widetilde{\ell}$ slowly varying at ∞. Let $c(x)$ be any positive function satisfying $\lim_{x\to\infty} x\widetilde{\ell}(c(x))/c^2(x) = 1$ which implies that $c(x) \sim x^{1/2}\ell^*(x)$, $x \to \infty$, for some ℓ^* slowly varying at ∞.
(b1) *If*

$$\lim_{x\to\infty} \mathbb{P}\{|\log(1-W)| > x\}(\ell^*(x))^2 = 0 \tag{5}$$

then, with $a_n = \sqrt{b_n}$, the limiting distribution of $\frac{L_n - b_n}{a_n}$ is standard normal.
(b2) *Assume that*

$$\mathbb{P}\{|\log(1-W)| > x\} \ \sim\ \ell(x), \quad x \to \infty, \tag{6}$$

for some ℓ slowly varying at ∞, and that

$$\lim_{x\to\infty} \mathbb{P}\{|\log(1-W)| > x\}(\ell^*(x))^2 = \infty.$$

Then, with $a_n := \mu^{-3/2} c(\log n)\psi(n)$, the limiting distribution of $\frac{L_n - b_n}{a_n}$ is standard normal.
(c) *Assume that*

$$\mathbb{P}\{|\log W| > x\} \ \sim\ x^{-\alpha}\widetilde{\ell}(x), \quad x \to \infty, \tag{7}$$

for some $\widetilde{\ell}$ slowly varying at ∞ and $\alpha \in (1,2)$. Let $c(x)$ be any positive function satisfying $\lim_{x\to\infty} x\widetilde{\ell}(c(x))/c^\alpha(x) = 1$ which implies that $c(x) \sim x^{1/\alpha}\ell^*(x)$, $x \to \infty$, for some ℓ^* slowly varying at ∞.
(c1) *If*

$$\lim_{x\to\infty} \mathbb{P}\{|\log(1-W)| > x\} x^{2/\alpha - 1}(\ell^*(x))^2 = 0, \tag{8}$$

then, with $a_n = \sqrt{b_n}$, the limiting distribution of $\frac{L_n - b_n}{a_n}$ is standard normal.
(c2) *Assume that*

$$\mathbb{P}\{|\log(1-W)| > x\} \ \sim\ x^{-\beta}\ell(x), \quad x \to \infty, \tag{9}$$

for some $\beta \in [0, 2/\alpha - 1]$ and some ℓ slowly varying at ∞. In the case $\beta = 2/\alpha - 1$ assume additionally that

$$\lim_{x\to\infty} \mathbb{P}\{|\log(1-W) > x|\} x^{2/\alpha - 1}(\ell^*(x))^2 = \infty.$$

Then

$$\frac{L_n - b_n}{\mu^{-1-1/\alpha} c(\log n)\psi(n)} \overset{d}{\to} \int_{[0,1]} v^{-\beta} \mathrm{d}Z(v),$$

where $(Z(v))_{v\in[0,1]}$ *is an* α*-stable Lévy process such that* $Z(1)$ *has characteristic function*

$$u \mapsto \exp\{-|u|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) + i\sin(\pi\alpha/2)\operatorname{sgn}(u))\},\ u \in \mathbb{R}. \tag{10}$$

Throughout one can take $b'_n := \mu^{-1}\int_{[0,\log n]} \mathbb{P}\{|\log(1-W)| > x\}\mathrm{d}x$ *in place of* b_n.

Remark 1.3. The integrals $\int_{[0,1]} v^{-\beta}\mathrm{d}Z(v)$ appearing in the theorem and also in formulae (34) and (35) are understood to be equal to $Z(1)$ in the case $\beta = 0$ and to be defined by integration by parts formula

$$\int_{[0,1]} v^{-\beta}\mathrm{d}Z(v) = Z(1) + \beta \int_{[0,1]} v^{-\beta-1} Z(v)\mathrm{d}v$$

in the case $\beta \in (0, 1/\alpha)$ (when referring to formula (34) we take $\alpha = 2$). Note that the latter is consistent with the standard definition of stochastic integrals (with respect to semimartingales). It is known that

$$\log \mathbb{E}\exp\left(\mathrm{i}t \int_{[0,1]} v^{-\beta}\mathrm{d}Z(v)\right) = \int_{[0,1]} \log \mathbb{E}\exp\left(\mathrm{i}tv^{-\beta} Z(1)\right)\mathrm{d}v, \quad t \in \mathbb{R},$$

from which it follows that the integral is indeed well-defined only if $\beta \in [0, 1/\alpha)$ and that

$$\int_{[0,1]} v^{-\beta}\mathrm{d}Z(v) \overset{d}{=} (1-\alpha\beta)^{-1/\alpha} Z(1).$$

Remark 1.4. Theorem 1.2 does not cover two interesting cases. Assume that the standing assumptions of the theorem hold.
Case (b3): Condition (4) holds, $\sigma^2 = \infty$, and

$$\mathbb{P}\{|\log(1-W)| > x\} \ \sim\ \frac{d}{(\ell^*(x))^2}, \quad x \to \infty,$$

for some $d > 0$ and $\ell^*(x)$ defined in part (b) of the theorem.
Case (c3): Condition (7) holds, and

$$\mathbb{P}\{|\log(1-W)| > x\} \ \sim\ \frac{dx^{1-2/\alpha}}{(\ell^*(x))^2}, \quad x \to \infty,$$

for some $d > 0$ and $\ell^*(x)$ defined in part (c) of the theorem.

Some partial results and discussion of the problems which arise in these cases can be found in Remark 3.7.

Remark 1.5. We conjecture that under the assumption $\mu < \infty$ the conditions given in Theorem 1.2 and Remark 1.4 are necessary and sufficient for the weak convergence of L_n, properly normalized and centered.

The rest of the paper is structured as follows. In Section 2 we point out the set of conditions under which the number of zero decrements of a nonincreasing Markov chain weakly converges to a geometric law (Theorem 2.1). Theorem 1.1 then follows as a particular case. Section 3 is devoted to proving Theorem 1.2. Some results derived in Section 3 can be used to answer a question asked in [19]. A detailed discussion of this is given in Section 4. Some auxiliary facts are collected in the Appendix.

2 Number of zero decrements of nonincreasing Markov chains

2.1 Definitions

With $M \in \mathbb{N}_0$ given and any $n \geq M$, $n \in \mathbb{N}$, let $I := \big(I_k(n)\big)_{k \in \mathbb{N}_0}$ be a *nonincreasing Markov chain* with $I_0(n) = n$, state space $\mathbb{N}$ and transition probabilities

$$\begin{aligned}
\mathbb{P}\{I_k(n) = j | I_{k-1}(n) = i\} &= \pi_{i,j}, \quad i \geq M+1 \text{ and either } M < j \leq i \text{ or } M = j < i, \\
\mathbb{P}\{I_k(n) = j | I_{k-1}(n) = i\} &= 0, \quad i < j, \\
\mathbb{P}\{I_k(n) = M | I_{k-1}(n) = M\} &= 1.
\end{aligned}$$

Denote by

$$Z_n := \#\big\{k \in \mathbb{N}_0 : I_k(n) - I_{k+1}(n) = 0, I_k(n) > M\big\}$$

the number of zero decrements of the Markov chain before the absorption. Assuming that, for every $M+1 \leq i \leq n$, $\pi_{i,\,i-1} > 0$, the absorption at state M is certain, and Z_n is a.s. finite.

Neglecting zero decrements of I along with renumbering of indices lead to a *decreasing Markov chain* $J := \big(J_k(n)\big)_{k \in \mathbb{N}_0}$ with $J_0(n) = n$ and transition probabilities

$$\widetilde{\pi}_{i,j} = \frac{\pi_{i,j}}{1 - \pi_{i,\,i}}, \quad i > j \geq M$$

(the other probabilities are the same as for I). The chain J visits a given state k and the chain I visits the state k for the first time with the same probability

$$g_{n,k} := \sum_{m \geq 0} \mathbb{P}\{J_m(n) = k\}, \quad k \leq n, k \in \mathbb{N}.$$

Note that $g_{n,n} = 1$ and that $g_{n,k}$ is the potential function of J.

Let $(R_j)_{M+1 \leq j \leq n}$ be independent random variables such that $R_j \overset{d}{=} \text{geom}(1 - \pi_{j,j})$. Assuming the R_j's independent of the sequence of states visited by J we may identify R_j with the time I spends in the state j provided this state is visited. With this at hand Z_n can be conveniently represented as

$$Z_n \overset{d}{=} \sum_{k \geq 0} R_{J_k(n)} 1_{\{J_k(n) > M\}}. \tag{11}$$

2.2 Main result of the section

Theorem 2.1 given below proves that the number of zero decrements of a nonincreasing Markov chain weakly converges to a geometric law whenever the probability of delay at the present state and that of transition to the absorption state are asymptotically balanced, and the Markov chain has no 'stationary' version. An interesting feature of this quite general result is that its proof needs nothing beyond simple distributional recurrence (16).

Theorem 2.1. *Assume that* $\lim_{n \to \infty} g_{n,k} = 0$ *for each* $k \in \mathbb{N}$, $\lim_{n \to \infty} \pi_{n,n} = 0$ *and*

$$\lim_{n \to \infty} \frac{\pi_{n,n}}{\pi_{n,M}} = c \in (0, \infty). \tag{12}$$

Then

$$Z_n \overset{d}{\to} Z \overset{d}{=} \text{geom}((c+1)^{-1}), \quad n \to \infty.$$

Theorem 2.1 will be proved by the method of moments. To this end, we have to possess some information about the moments of integer orders of the limiting geometric law. The explicit expressions are complicated and actually not needed. The moments satisfy a simple recurrence which is sufficient for our needs.

Lemma 2.2. *Let* $X \stackrel{d}{=} \text{geom}(a)$, $a > 0$. *The moments* $m_k := \mathbb{E}X^k$, $k \in \mathbb{N}$ *can be recursively obtained via*

$$m_1 = b, \quad m_j = b\Big(1 + \sum_{i=1}^{j-1} \binom{j}{i} m_i\Big), \quad j = 2, 3, \ldots, \tag{13}$$

where $b := (1-a)/a$.

Proof. Let $(\zeta_k)_{k\in\mathbb{N}}$ be independent Bernoulli random variables with success probability a. Then

$$X \stackrel{d}{=} \inf\{k \in \mathbb{N} : \zeta_k = 1\} - 1 = 1_{\{\zeta_1 = 0\}}\big(1 + (\inf\{k \in \mathbb{N}\backslash\{1\} : \zeta_k = 1\} - 1)\big) =: 1_{\{\zeta_1=0\}}(1 + X'),$$

where X' is independent of ζ_1 and $X' \stackrel{d}{=} X$. The latter implies

$$\mathbb{E}X^j = (1-a)\mathbb{E}(1+X)^j, \quad j \in \mathbb{N},$$

and representation (13) follows. □

Now we are ready to prove Theorem 2.1. For notational convenience we assume that $M = 0$. For other M's the argument is the same.

Let V_n denote the size of the last decrement. Then

$$\mathbb{P}\{V_n = k\} = g_{n,k}\widetilde{\pi}_{k,0} = g_{n,k}\frac{\pi_{k,0}}{1 - \pi_{k,k}}, \quad k = 1, 2, \ldots, n, \tag{14}$$

and

$$\lim_{n\to\infty} \mathbb{P}\{V_n = k\} = 0, \quad \text{for each } k \in \mathbb{N}. \tag{15}$$

Since the geometric law is uniquely determined by its moments, it suffices to prove that, for each $i \in \mathbb{N}$, $\lim_{n\to\infty} \mathbb{E}Z_n^i = \mathbb{E}Z^i$. To this end, we will use the induction on i and start with the case $i = 1$. Using representation (11) and conditioning on the first decrement of J we deduce the distributional equality

$$Z_n \stackrel{d}{=} \widehat{Z}_{J(n)} + R_n, \tag{16}$$

where, for each $k \in \mathbb{N}$, $\widehat{Z}_k$ is independent of both $J(n) := J_1(n)$ and R_n, and has the same law as Z_k. Equality (16) (or just (11)) implies that

$$\mathbb{E}Z_n \;=\; \sum_{k=1}^{n} g_{n,k}\mathbb{E}R_k \stackrel{(14)}{=} \sum_{k=1}^{n} \mathbb{P}\{V_n = k\}\frac{\pi_{k,k}}{\pi_{k,0}}$$

Recalling (15) and (12) and applying Lemma 5.1 to the last sum lead to the conclusion $\lim_{n\to\infty} \mathbb{E}Z_n = c = \mathbb{E}Z$.

Assume now that $\lim_{n\to\infty} \mathbb{E}Z_n^i = \mathbb{E}Z^i$ for all $i \le j-1$, $i \in \mathbb{N}$. We have to prove that $\lim_{n\to\infty} \mathbb{E}Z_n^j = \mathbb{E}Z^j$. In view of Lemma 2.2 it suffices to check that $\lim_{n\to\infty} \mathbb{E}Z_n^j = m_j$, where m_j satisfies (13) with $b = c$ and $m_i = \mathbb{E}Z^i$. Using (16) yields

$$\mathbb{E}Z_n^j = \mathbb{E}Z_{J(n)}^j + \sum_{i=0}^{j-1} \binom{j}{i} \mathbb{E}Z_{J(n)}^i \mathbb{E}R_n^{j-i} =: \mathbb{E}Z_{J(n)}^j + b_n,$$

or, equivalently,

$$\mathbb{E}Z_n^j = \sum_{k=1}^{n} g_{n,\,k} b_k = \sum_{k=1}^{n} \mathbb{P}\{V_n = k\} \frac{1-\pi_{k,\,k}}{\pi_{k,\,0}} b_k.$$

In view of Lemma 5.1 to finish the proof it remains to show that

$$\lim_{n\to\infty} \frac{1-\pi_{n,\,n}}{\pi_{n,\,0}} b_n = c\left(1 + \sum_{i=1}^{j-1} \binom{j}{i} m_i\right)$$

or equivalently that, for $i \le j-1$,

$$\lim_{n\to\infty} \frac{1-\pi_{n,\,n}}{\pi_{n,\,0}} \mathbb{E}Z_{J(n)}^i \mathbb{E}R_n^{j-i} = c\,\mathbb{E}Z^i \quad \text{and} \quad \lim_{n\to\infty} \frac{1-\pi_{n,\,n}}{\pi_{n,\,0}} \mathbb{E}R_n^j = c. \tag{17}$$

Applying Lemma 2.2 with $a = 1-\pi_{n,\,n}$ to the R_n's we conclude that

$$\mathbb{E}R_n^{j-i} \ \sim\ \mathbb{E}R_n \ \sim\ \pi_{n,\,n}, \quad n\to\infty.$$

Further, one can easily check that $\lim_{n\to\infty} b_n = 0$, hence

$$\lim_{n\to\infty} \mathbb{E}Z_{J(n)}^i = \mathbb{E}Z^i, \quad i \le j-1.$$

These two observations immediately establish (17). The proof is complete.

2.3 Proof of Theorem 1.1

In this subsection we prove Theorem 1.1 by an application of Theorem 2.1. To distinguish general (nonincreasing) Markov chains in the previous subsections from the particular Markov chain discussed below we mark all the quantities which correspond to the latter with asterisk, for instance, $I \to I^*$, $g_{n,k} \to g^*_{n,k}$ etc.

Now we present one more construction of the Bernoulli sieve which highlights the connection with nonincreasing Markov chains. The Bernoulli sieve can be realized as a random occupancy scheme in which n 'balls' are allocated over an infinite array of 'boxes' indexed $1, 2, \ldots$ according to the following rule. At the first round each of n balls is dropped in box 1 with probability W_1. At the second round each of the remaining balls is dropped in box 2 with probability W_2, and so on. The procedure proceeds until all n balls get allocated. Let $I^*_k(n)$ denote the number of remaining balls (out of n) after the kth round. Then $I^* := (I^*_k(n))_{k\in\mathbb{N}_0}$ is a pattern of nonincreasing Markov chains described in Subsection 2.1 with $M = 0$ and

$$\pi^*_{i,j} = \binom{i}{j} \mathbb{E}W^j(1-W)^{i-j}, \quad j \le i. \tag{18}$$

It is plain that L_n is the number of zero decrements of I^* before the absorption.

The assumption $\mu = \infty$ implies that $\lim_{n\to\infty} g^*_{n,k} = 0$, for each $k \in \mathbb{N}$. In the case that the law of $|\log W|$ is nonlattice this fact was pointed out in formula (16) in [13]. The complementary case does not require any new proof once one has noticed that the overshoot at point x of a standard random walk diverges to $+\infty$ in probability (as $x\to\infty$) under the sole assumption that the step of the random walk has infinite mean. In view of (18) $\pi^*_{n,n} = \mathbb{E}W^n \to 0$, as $n\to\infty$ (recall that the law of W has no atom at 1), and condition (12) reduces to (1). According to Theorem 2.1, relation (2) holds.

Condition (3) is equivalent to

$$\lim_{x\downarrow 0}\frac{\mathbb{P}\{|\log W|\le x\}}{\mathbb{P}\{|\log(1-W)|\le x\}}=c.$$

Applying Lemma 5.2 with $\xi=|\log W|$ and $\eta=|\log(1-W)|$ establishes implication (3) $\Rightarrow$ (1), thereby completing the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In the first (main) part of the proof we work with a poissonized version of the Bernoulli sieve. Specifically we assume that the balls are thrown at arrival times $(\tau_n)_{n\in\mathbb{N}}$ of a unit rate Poisson process $(\pi_t)_{t\ge 0}$. The quantity in focus is then $L(t):=L_{\pi_t}$, where (π_t) is independent of (L_j). At the last step of the proof we return to the original, fixed n problem (this step is called *depoissonization*) and prove the implication

$$\frac{L(t)-b(t)}{a(t)}\overset{d}{\to}X,\quad t\to\infty\Rightarrow\frac{L_n-b(n)}{a(n)}\overset{d}{\to}X,\quad n\to\infty.$$

Set

$$S_k:=-\log T_k=|\log W_1|+\ldots+|\log W_k|\quad\text{and}\quad \widehat{S}_k:=\widehat{S}_0+S_k,\quad k\in\mathbb{N}_0,$$

where $\widehat{S}_0$ is a random variable which is independent of (S_k) and has distribution

$$\mathbb{P}\{\widehat{S}_0\le x\}=\mu^{-1}\int_{[0,x]}\mathbb{P}\{|\log W|>y\}\mathrm{d}y,\quad x\ge 0.$$

Define

$$N(x):=\inf\{k\in\mathbb{N}_0:S_k>x\}=\#\{k\in\mathbb{N}_0:S_k\le x\}\ \text{ and }\ \widehat{N}(x):=\#\{n\in\mathbb{N}_0:\widehat{S}_n\le x\},\ \ x\ge 0,$$

and recall that $(N(x))_{x\ge 0}$ and $(\widehat{N}(x))_{x\ge 0}$ are non-stationary and stationary renewal processes, respectively. For later use, we recall (see p. 55 in [15] for the proof) that $N(x)$ enjoys the following (distributional) subadditivity property

$$N(x+y)-N(x)\overset{d}{\le}N(y),\quad x,y\ge 0. \tag{19}$$

In the sequel we work with the following random variables

$$C(t):=\sum_{k\ge 0}\varphi(t-S_k)1_{\{S_k\le t\}}=\int_{[0,\,t]}\varphi(t-x)\mathrm{d}N(x),\quad t\ge 0$$

and

$$\widehat{C}(t):=\sum_{k\ge 0}\varphi(t-\widehat{S}_k)1_{\{\widehat{S}_k\le t\}}=\int_{[0,\,t]}\varphi(t-x)\mathrm{d}\widehat{N}(x),\quad t\ge 0,$$

where $\varphi(t):=\psi(e^t)$, $t\in\mathbb{R}$, $\psi(t):=\mathbb{E}e^{-t(1-W)}$, $t\ge 0$.

We show in Lemma 3.1 that convergence in distribution of $L(t)$ is completely determined by convergence in distribution of

$$L^*(t):=\sum_{k\ge 1}\exp(-te^{-S_{k-1}}(1-W_k))1_{\{S_{k-1}\le\log t\}}.$$

The Bernoulli sieve is governed by two sources of randomness: randomness of the 'environment' (W_k) and sampling variability (i.e. the variability of the occupancy scheme with *deterministic* frequencies obtained by conditioning on (W_k)). Since $L^*(t)$ is a function of the environment alone we conclude that the weak convergence of $L(t)$ (L_n) is completely determined by the randomness of the environment, whereas the influence of the sampling variability is negligible.

In its turn convergence in distribution of $L^*(t)$ is determined either by that of $L^*(t)-C(\log t)$ or that of $C(\log t)$, or that of both, and our main task is to find out what is the extent of their interplay. In the cases (a), (b1) and (c1) the contribution of $L^*(t)-C(\log t)$ dominates, whereas in the cases (b2) and (c2) it is negligible in comparison with the contribution of $C(\log t)$.[1]

We divide the proof of the theorem into several steps.

STEP 1. The purpose of this step is proving a central limit theorem for $L(t)-C(\log t)$ (Lemma 3.3). To this end we first show that the asymptotic behavior of $L(t)$ coincides with that of $L^*(t)$.

In what follows we write that the family of random variables is tight meaning that the family of laws of these random variables is tight.

Lemma 3.1. *Whenever $\mu<\infty$ and the law of $|\log W|$ is non-lattice, the families $\big(L(t)-L^*(t)\big)_{t\geq 1}$ and $\big(C(t)-\widehat{C}(t)\big)_{t\geq 0}$ are tight.*

Proof. Set $M(t):=M_{\pi_t}$ and $K(t):=K_{\pi_t}$. These are the index of the last occupied box and the number of occupied boxes in the poissonized version of the Bernoulli sieve, respectively. Clearly, $L(t)=M(t)-K(t)$.

FACT 1: The family $\big(M(t)-N(\log t)\big)_{t\geq 1}$ is tight.

We use the representation $M(t)=N(|\log U_{1,\pi_t}|)$, where $U_{1,n}:=\min\limits_{1\leq j\leq n} U_j$. It is well-known that $|\log U_{1,n}|-\log n \overset{d}{\to} G$, $n\to\infty$, where G is a random variable with the standard Gumbel distribution. Since (π_t) is independent of $U_{1,n}$ we also have $|\log U_{1,\pi_t}|-\log \pi_t \overset{d}{\to} G$, $t\to\infty$. By noting that $\log\pi_t-\log t\overset{P}{\to} 0$, $t\to\infty$ we finally conclude that $|\log U_{1,\pi_t}|-\log t\overset{d}{\to} G$, $t\to\infty$. Using (19) along with independence of $\big(N(x)\big)$ and $U_{1,\,\pi_t}$ we obtain

$$\begin{aligned}
M(t)-N(\log t) &\leq \big(N(|\log U_{1,\,\pi_t}|)-N(\log t)\big)\big)1_{\{|\log U_{1,\,\pi_t}|\geq \log t\}}\\
&\overset{d}{\leq} N(|\log U_{1,\,\pi_t}|-\log t)1_{\{|\log U_{1,\,\pi_t}|\geq \log t\}} \overset{d}{\to} N(G)1_{\{G\geq 0\}},\quad t\to\infty.
\end{aligned}$$

Similarly

$$\begin{aligned}
M(t)-N(\log t) &\geq -\big(N(\log t)-N(|\log U_{1,\pi_t}|)\big)1_{\{|\log U_{1,\pi_t}|<\log t\}}\\
&\overset{d}{\geq} -N\big(\log t-|\log U_{1,\pi_t}|\big)1_{\{|\log U_{1,\pi_t}|<\log t\}} \overset{d}{\to} -N(-G)1_{\{G<0\}},\quad t\to\infty.
\end{aligned}$$

FACT 2: The family $\big(K(t)-\mathbb{E}(K(t)|(W_k))\big)_{t\geq 0}$ is tight.

This was proved in formula (28) in [11].

FACT 3: The family

$$\Big(L(t)-N(\log t)+\sum_{k\geq 1}\big(1-\exp(-te^{-S_{k-1}}(1-W_k))\big)\Big)_{t\geq 1}$$

is tight.

[1] It seems that there are situations (cases (b3) and (c3) introduced in Remark 1.4) when contributions of both variables are significant, and both of these determine the asymptotics of $L(t)$. See Remark 1.4 and Remark 3.7 for more details.

Since

$$\mathbb{E}(K(t)|(W_k)) = \sum_{k\geq 1}\big(1-e^{-tP_k}\big) = \sum_{k\geq 1}\big(1-\exp(-te^{-S_{k-1}}(1-W_k))\big),$$

Fact 1 and Fact 2 together imply the statement.
FACT 4: The family $\big(Y(t)\big)_{t\geq 1}$, where

$$Y(t) := \sum_{k\geq 1}\big(1-\exp(-te^{-S_{k-1}}(1-W_k))\big)1_{\{S_{k-1}>\log t\}},$$

is tight.

Since $1-\varphi$ is monotone and integrable on $(-\infty,0]$, it is directly Riemann integrable on $(-\infty,0]$. Hence, by the key renewal theorem (see Theorem 4.2 in [1])

$$\mathbb{E}Y(e^t) = \mathbb{E}\sum_{k\geq 0}\big(1-\varphi(t-S_k)\big)1_{\{S_k>t\}} \ \to\ \mu^{-1}\int_{[0,1]}\big(1-\psi(y)\big)y^{-1}\mathrm{d}y<\infty,$$

and Fact 4 follows.

Now we are ready to prove the lemma. Since

$$L(t)-L^*(t) = \bigg(L(t)-N(\log t)+\sum_{k\geq 1}\big(1-\exp(-te^{-S_{k-1}}(1-W_k))\big)\bigg)-Y(t),$$

the first assertion of the lemma follows from Fact 3 and Fact 4.

In view of the inequality

$$\begin{aligned}
-\Big(\varphi(t-\widehat{S}_k)-\varphi(t-S_k)\Big)1_{\{\widehat{S}_k\leq t\}} &\leq \varphi(t-S_k)1_{\{S_k\leq t\}}-\varphi(t-\widehat{S}_k)1_{\{\widehat{S}_k\leq t\}}\\
&= \varphi(t-S_k)1_{\{S_k\leq t<\widehat{S}_k\}}\\
&- \Big(\varphi(t-\widehat{S}_k)-\varphi(t-S_k)\Big)1_{\{\widehat{S}_k\leq t\}}\\
&\leq \varphi(t-S_k)1_{\{S_k\leq t<\widehat{S}_k\}}\\
&= \varphi(t-S_k)1_{\{S_k\leq t,\widehat{S}_0>t\}}\\
&+ \varphi(t-S_k)1_{\{t-\widehat{S}_0<S_k\leq t,\widehat{S}_0\leq t\}} \ \text{ a.s.},
\end{aligned}$$

to prove the second assertion it suffices to check the tightness of

$$\mathcal{C}_1 := \bigg(\sum_{k\geq 0}\varphi(t-S_k)1_{\{S_k\leq t<\widehat{S}_k\}}\bigg)_{t\geq 0} \ \text{ and } \ \mathcal{C}_2 := \bigg(\sum_{k\geq 0}\Big(\varphi(t-\widehat{S}_k)-\varphi(t-S_k)\Big)1_{\{\widehat{S}_k\leq t\}}\bigg)_{t\geq 0}.$$

Using (19) gives

$$\sum_{k\geq 0}\varphi(t-S_k)1_{\{t-\widehat{S}_0<S_k\leq t,\,\widehat{S}_0\leq t\}}\leq \varphi(0)\big(N(t)-N(t-\widehat{S}_0)\big)1_{\{\widehat{S}_0\leq t\}}\overset{d}{\leq}\varphi(0)N(\widehat{S}_0).$$

It is clear that

$$\bigg(\sum_{k\geq 0}\varphi(t-S_k)1_{\{S_k\leq t\}}\bigg)1_{\{\widehat{S}_0>t\}} \ \overset{P}{\to}\ 0,\quad t\to\infty,$$

and the tightness of $\mathcal{C}_1$ follows. Using the mean value theorem for differentiable functions and the monotonicity of ψ' we obtain

$$\Big(\varphi(t-\widehat{S}_k)-\varphi(t-S_k)\Big)1_{\{\widehat{S}_k\le t\}}\le e^{t-S_k}(-\psi'(e^{t-\widehat{S}_k}))1_{\{\widehat{S}_k\le t\}}\widehat{S}_0=-\varphi'(t-\widehat{S}_k)1_{\{\widehat{S}_k\le t\}}\widehat{S}_0e^{\widehat{S}_0}.$$

Since

$$\mathbb{E}\sum_{k\ge 0}(-\varphi'(t-\widehat{S}_k))1_{\{\widehat{S}_k\le t\}}=\mu^{-1}\int_{[0,t]}(-\varphi'(y))\mathrm{d}y\ \to\ \mu^{-1}\varphi(0),\quad t\to\infty,$$

the family $\mathcal{C}_2$ is tight. The proof is complete. $\square$

Further we need a preliminary result which establishes a weak law of large numbers for $C(t)$.

Lemma 3.2. *Suppose $\mu<\infty$, $\nu=\infty$, and the distribution of $|\log W|$ is non-lattice. Then*

$$\frac{C(t)}{k(t)}\ \overset{P}{\to}\ \mu^{-1},\quad t\to\infty,$$

where

$$k(x):=\int_0^x\varphi(y)\mathrm{d}y,\quad x>0.$$

Proof. The assumption $\nu=\infty$ is equivalent to $\lim_{x\to\infty}k(x)=\infty$. In view of Chebyshev's inequality it is enough to check that

$$\mathbb{E}C^2(t)\ \sim\ (\mathbb{E}C(t))^2\ \sim\ \mu^{-2}k^2(t),\quad t\to\infty.$$

By Lemma 5.4(a), the required asymptotics of $\mathbb{E}C(t)$ follows easily. Using the equality

$$C(t)=\varphi(t)+C'(t-S_1)1_{\{S_1\le t\}}\ \text{ a.s.},$$

where $C'(t):=\sum_{k\ge1}\varphi(t-S_k+S_1)1_{\{S_k-S_1\le t\}}\overset{d}{=}C(t)$ is independent of S_1, we have

$$\mathbb{E}C^2(t)=2\int_{[0,\,t]}\varphi(t-x)\mathbb{E}C(t-x)\mathrm{d}\mathbb{E}N(x)-\int_{[0,\,t]}\varphi^2(t-x)\mathrm{d}\mathbb{E}N(x).\tag{20}$$

The second term exhibits the following asymptotics

$$\int_{[0,\,t]}\varphi^2(t-x)\mathrm{d}\mathbb{E}N(x)=o(k(t)),\quad t\to\infty.\tag{21}$$

To see this, use the key renewal theorem in the case $\int_{[0,\,\infty)}\varphi^2(x)\mathrm{d}x<\infty$ or Lemma 5.4(a) followed by l'Hôpital rule in the case $\lim_{t\to\infty}\int_{[0,\,t]}\varphi^2(x)\mathrm{d}x=\infty$.

Since both $k(t)$ and $(1-\varphi(t))k(t)$ are nondecreasing functions we apply Lemma 5.4(b) to obtain, as $t\to\infty$,

$$\int_{[0,\,t]}\varphi(t-x)k(t-x)\mathrm{d}\mathbb{E}N(x)\ \sim\ \mu^{-1}\int_{[0,\,t]}\varphi(x)k(x)\mathrm{d}x=(2\mu)^{-1}k^2(t).\tag{22}$$

Further, for fixed $a\in(0,t)$

$$\int_{[t-a,\,t]}\varphi(t-x)k(t-x)\mathrm{d}\mathbb{E}N(x)\le k(a)\big(\mathbb{E}N(t)-\mathbb{E}N(t-a)\big)\le k(a)\mathbb{E}N(a)\tag{23}$$

in view of (19). Hence

$$\int_{[0,\,t]} \varphi(t-x)k(t-x)\mathrm{d}\mathbb{E}N(x) \;\sim\; \int_{[0,\,t-a]} \varphi(t-x)k(t-x)\mathrm{d}\mathbb{E}N(x), \quad t\to\infty.$$

Likewise, since

$$\sup_{x\in[0,\,a]} \mathbb{E}C(x)<\infty,$$

we conclude that

$$\int_{[t-a,\,t]} \varphi(t-x)\mathbb{E}C(t-x)\mathrm{d}\mathbb{E}N(x) \le \sup_{x\in[0,a]} \mathbb{E}C(x)\mathbb{E}N(a)<\infty. \tag{24}$$

Now we are ready to derive the asymptotics of $\mathbb{E}C^2(t)$. For any $\varepsilon\in(0,\mu^{-1})$ there exists $x_0>0$ such that $\mu^{-1}-\varepsilon\le \mathbb{E}C(y)/k(y)\le \mu^{-1}+\varepsilon$ for $y\ge x_0$. With this x_0 we have

$$\begin{aligned}
\int_{[0,\,t]} \varphi(t-x)\mathbb{E}C(t-x)\mathrm{d}\mathbb{E}N(x) \quad &\le \quad (\mu^{-1}+\varepsilon)\int_{[0,\,t-x_0]} \varphi(t-x)k(t-x)\mathrm{d}\mathbb{E}N(x)\\
&+ \quad \int_{[t-x_0,\,t]} \varphi(t-x)\mathbb{E}C(t-x)\mathrm{d}\mathbb{E}N(x)\\
&\overset{(23),(24)}{\sim} \quad (\mu^{-1}+\varepsilon)\int_{[0,\,t]} \varphi(t-x)k(t-x)\mathrm{d}\mathbb{E}N(x)+O(1)\\
&\overset{(22)}{\sim} \quad (\mu^{-1}+\varepsilon)(2\mu)^{-1}k^2(t).
\end{aligned}$$

Sending $\varepsilon\to 0$ and recalling (20) and (21) we conclude that

$$\limsup_{t\to\infty}\frac{\mathbb{E}C^2(t)}{k^2(t)}\le \mu^{-2}.$$

Arguing similarly we obtain the converse inequality for the lower limit. The proof is complete. □

Lemma 3.3. *Suppose $\mu<\infty$, $\nu=\infty$, and the distribution of $|\log W|$ is non-lattice. Then*

$$\frac{L(t)-C(\log t)}{\sqrt{\mu^{-1}k(\log t)}} \;\overset{d}{\to}\; \mathcal{N}(0,1), \quad t\to\infty.$$

Proof. By Lemma 3.1, it is enough to prove that

$$\frac{L^*(t)-C(\log t)}{\sqrt{\mu^{-1}k(\log t)}} \;\overset{d}{\to}\; \mathcal{N}(0,1), \quad t\to\infty. \tag{25}$$

Set

$$X_{ti}:=\frac{\big(\exp\big(-te^{-S_{i-1}}(1-W_i)\big)-\psi(te^{-S_{i-1}})\big)1_{\{S_{i-1}\le\log t\}}}{\sqrt{\mu^{-1}k(\log t)}}, \quad i\in\mathbb{N},\ t>1,$$

and note that $\mathbb{E}\big(X_{ti}|(W_k)_{k\le i-1}\big)=0$. By a martingale central limit theorem (Corollary 3.1 in [16]), relation

$$\frac{L^*(n)-C(\log n)}{\sqrt{\mu^{-1}k(\log n)}} \;\overset{d}{\to}\; \mathcal{N}(0,1), \quad n\to\infty, \tag{26}$$

which is just (25) with continuous variable t replaced by integer n, will hold once we can show that

$$\sum_{i\geq 1}\mathbb{E}(X_{ni}^2|(W_k)_{k\leq i-1}) \overset{P}{\to} 1, \quad n\to\infty, \tag{27}$$

and that, for all $\varepsilon>0$,

$$\sum_{i\geq 1}\mathbb{E}(X_{ni}^2 1_{\{|X_{ni}|>\varepsilon\}}|(W_k)_{k\leq i-1}) \overset{P}{\to} 0, \quad n\to\infty. \tag{28}$$

It suffices to establish (27), as, in view of $|X_{ni}|\leq 1/\sqrt{\mu^{-1}k(\log n)}$, (28) will follow from it.

We have

$$\begin{aligned}\sum_{i\geq 1}\mathbb{E}(X_{e^t i}^2|(W_k)_{k\leq i-1}) &= \frac{\int_{[0,t]}\big(\psi(2e^{t-x})-\varphi^2(t-x)\big)\mathrm{d}N(x)}{\mu^{-1}k(t)}\\ &= \frac{C(t)}{\mu^{-1}k(t)}-\frac{\int_{[0,t]}\big(\psi(e^{t-x})-\psi(2e^{t-x})\big)\mathrm{d}N(x)}{\mu^{-1}k(t)}-\frac{\int_{[0,t]}\varphi^2(t-x)\mathrm{d}N(x)}{\mu^{-1}k(t)}.\end{aligned}$$

By Lemma 3.2, $\lim_{t\to\infty} C(t)/(\mu^{-1}k(t))=1$ in probability. To complete the proof of (27) one has to check that the second and the third terms converge to zero in probability. For the third this follows from (21) and Markov's inequality. The function $t\mapsto\psi(e^t)-\psi(2e^t)$ is directly Riemann integrable on $\mathbb{R}$ since it is nonnegative and integrable, and the function $t\mapsto e^{-t}\big(\psi(e^t)-\psi(2e^t)\big)$ is nonincreasing (see, for instance, the proof of Corollary 2.17 in [6]). By the key renewal theorem

$$\begin{aligned}\mathbb{E}\int_{[0,t]}\big(\psi(e^{t-x})-\psi(2e^{t-x})\big)\mathrm{d}N(x) &\leq \mathbb{E}\int_{[0,\infty)}\big(\psi(e^{t-x})-\psi(2e^{t-x})\big)\mathrm{d}N(x)\\ &\to \mu^{-1}\mathbb{E}\int_{[0,\infty]}\big(e^{-y(1-W)}-e^{-2y(1-W)}\big)y^{-1}\mathrm{d}y\\ &= \mu^{-1}\log 2, \quad t\to\infty,\end{aligned}$$

which proves the required result for the second term.

It remains to pass from (26) to (25). We first note that the function $k(\log t)$ is slowly varying at ∞. This follows from the equality $k(\log t)=\int_{[1,\,t]}\psi(y)y^{-1}\mathrm{d}y$ and the representation theorem for slowly varying functions (Theorem 1.3.1 in [4]). To prove that $\lim_{t\to\infty}\frac{k(\log t)}{k(\log[t])}=1$, where $[t]$ denotes the integer part of t, use the slow variation of $k(\log t)$ together with the monotonicity to conclude

$$1\leq\frac{k(\log t)}{k(\log[t])}\leq\frac{k(\log([t]+1))}{k(\log[t])} \to 1, \quad t\to\infty.$$

Now we intend to prove the tightness of the family $\big(C(t)-C([t])\big)$. To this end we use the equality

$$C([t])-C(t)=\int_{[0,[t]]}\big(\varphi([t]-x)-\varphi(t-x)\big)\mathrm{d}N(x)-\int_{[[t],t]}\varphi(t-x)\mathrm{d}N(x).$$

By the mean value theorem

$$\begin{aligned}\varphi([t]-x)-\varphi(t-x) &= -\varphi'(\theta)(t-[t])\leq e^{\theta}(-\psi'(\theta))\\ &\leq e^{t-x}(-\psi'([t]-x))=-\varphi'([t]-x)e^{t-[t]}\leq -\varphi'([t]-x)e,\end{aligned}$$

where θ is some value from $[[t]-x, t-x]$. Consequently,

$$C([t]) - C(t) \le e \int_{[0,[t]]} (-\varphi'([t]-x)) \mathrm{d}N(x),$$

and the right-hand side is bounded in probability as, by the key renewal theorem, its expectation goes to $e\psi(1)\mu^{-1}$ (the function $t \mapsto -\varphi'(t)$ is directly Riemann integrable on $\mathbb{R}^+$ since it is integrable on $\mathbb{R}^+$ and nonnegative, and $t \mapsto e^{-t}(-\varphi'(t)) = -\psi'(e^t)$ is a nonincreasing function). On the other hand, in view of (19)

$$C([t]) - C(t) \ge -\int_{[[t],t]} \varphi(t-x)\mathrm{d}N(x) \ge -(N(t) - N([t])) \overset{d}{\ge} -N(1).$$

Finally we want to show that the family $\big(L^*(t) - L^*([t])\big)$ is tight. By Lemma 3.1, it is enough to check the tightness of $\big(L(t) - L([t])\big)$. Since $L(t) - L([t])$ represents the fluctuation of the number of empty boxes after throwing $\pi_t - \pi_{[t]}$ balls, and the latter variable is bounded from above by a Poisson variable with mean one, the desired tightness follows. The proof of the lemma is complete. $\square$

STEP 2. The purpose of this step is investigating convergence in distribution of $C(t)$. The cases (a), (b1) and (c1) and the cases (b2) and (c2) are treated separately in Lemma 3.4 and Lemma 3.6, respectively.

In what follows we use the following notation. If $\sigma^2 < \infty$ we denote by $Z(\cdot)$ the Brownian motion and set $g(t) := \sqrt{\sigma^2\mu^{-3}t}$. If condition (4) holds we denote by $Z(\cdot)$ the Brownian motion and let $g(t)$ be any nondecreasing function such that $g(t) \sim \mu^{-3/2}c(t)$, $t \to \infty$. If condition (7) holds we denote by $Z(\cdot)$ the α-stable Lévy process such that $Z(1)$ has characteristic function (10), and let $g(t)$ be any nondecreasing function such that $g(t) \sim \mu^{-1-1/\alpha}c(t)$, $t \to \infty$.

It is well-known that under either of the conditions of the preceding paragraph, i.e. whenever the law of $|\log W|$ belongs to the domain of attraction of an α-stable law, $\alpha \in (1, 2]$,

$$\frac{S_{[t\cdot]} - \mu(t\cdot)}{\mathrm{const}\, g(t)} \ \Rightarrow\ -Z(\cdot), \quad t \to \infty$$

in $D := D[0, 1]$ under the J_1 topology. While the one-dimensional convergence is a classical result [8], the functional version is due to Skorohod (Theorem 2.7 in [22]). Since

$$\sup_{u\in[0,1]} |\widehat{S}_{[tu]} - S_{[tu]}| = \widehat{S}_0,$$

the same functional limit theorem as above also holds for $\widehat{S}_{[t\cdot]}$. An appeal to Theorem 13.7.1 in [23] allows us to conclude that[2]

$$W_t(\cdot) := \frac{\widehat{N}(t\cdot) - \mu^{-1}(t\cdot)}{g(t)} \ \Rightarrow\ Z(\cdot), \quad t \to \infty, \tag{29}$$

in D under the M_1-topology. Certainly, (29) entails the one-dimensional convergence $W_t(1) \Rightarrow Z(1)$, $t \to \infty$. Hence, by Skorohod's representation theorem there exist versions $\bar{W}_t(1) \overset{d}{=} W_t(1)$ and $\bar{Z}(1) \overset{d}{=} Z(1)$ such that

$$\bar{W}_t(1) \ \to\ \bar{Z}(1), \quad t \to \infty,$$

[2]According to Theorem 1b in [3], relation (29) also holds for the non-stationary renewal process $(N(t))_{t\ge0}$. Since our argument imitates one given in [3], we omit details.

almost surely. In particular, for any $\varepsilon > 0$ there exists an a.s. finite $T > 0$ such that

$$|\bar{W}_v(1)| \leq |\bar{Z}(1)| + \varepsilon \quad \text{for all} \quad v \geq T. \tag{30}$$

For multiple later use let us write the following estimate: for any positive $x(t)$ such that $\lim_{t\to\infty} x(t) = \infty$ and any $a > 0$

$$\frac{\left| \int_{[0,\,at]} (\widehat{N}(v) - \mu^{-1}v)\mathrm{d}(-\varphi(v)) \right|}{x(t)} \overset{d}{\leq} o_P(1) + (|\bar{Z}(1)| + \varepsilon) \frac{\int_{[0,\,at]} g(v)\mathrm{d}(-\varphi(v))}{x(t)}, \tag{31}$$

where $o_P(1)$ denotes a term that converges to zero in probability, as $t \to \infty$. This can be proved as follows:

$$\begin{aligned}
\int_{[0,\,at]} (\widehat{N}(v) - \mu^{-1}v)\mathrm{d}(-\varphi(v)) &\overset{d}{=} \int_{[0,\,at]} \bar{W}_v(1) g(v)\mathrm{d}(-\varphi(v)) \\
&= \int_{[0,\,at]} \ldots 1_{\{T > at\}} + \int_{[0,\,T]} \ldots 1_{\{T \leq at\}} + \int_{[T,\,at]} \ldots 1_{\{T \leq at\}} \\
&=: I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

It is plain that $\lim_{t\to\infty} I_1(t) = 0$ in probability. As to $I_2(t)$, write

$$\frac{|I_2(t)|}{x(t)} \leq \frac{\int_{[0,\,T]} \left|\bar{W}_v(1)\right| g(v)\mathrm{d}(-\varphi(v))}{x(t)} \overset{P}{\to} 0, \quad t \to \infty.$$

Finally

$$\frac{|I_3(t)|}{x(t)} \leq \frac{\int_{[T,\,at]} \left|\bar{W}_v(1)\right| g(v)\mathrm{d}(-\varphi(v))}{x(t)} 1_{\{T \leq at\}} \overset{(30)}{\leq} (|\bar{Z}(1)| + \varepsilon) \frac{\int_{[0,\,at]} g(v)\mathrm{d}(-\varphi(v))}{x(t)}.$$

Lemma 3.4. *Let the assumptions of parts (a) or (b1), or (c1) of Theorem 1.2 hold. Then*

$$\frac{C(\log t) - \mu^{-1}k(\log t)}{\sqrt{k(\log t)}} \overset{P}{\to} 0, \quad t \to \infty, \tag{32}$$

and

$$\frac{L(t) - \mu^{-1}k(\log t)}{\sqrt{\mu^{-1}k(\log t)}} \overset{d}{\to} \mathcal{N}(0,1), \quad t \to \infty. \tag{33}$$

Proof. We start by noting that relation (33) is an immediate consequence of (32) and Lemma 3.3. By Lemma 3.1 relation (32) is equivalent to

$$\frac{\widehat{C}(t) - \mu^{-1}k(t)}{\sqrt{k(t)}} \overset{P}{\to} 0, \quad t \to \infty.$$

To prove it, we represent the latter ratio in a more convenient form

$$\begin{aligned}
\frac{\widehat{C}(t)-\mu^{-1}k(t)}{\sqrt{k(t)}} &= \frac{\int_{[0,t]}\varphi(t-v)\mathrm{d}\widehat{N}(v)-\mu^{-1}k(t)}{\sqrt{k(t)}} \\
&\overset{d}{=} \frac{\int_{[0,t]}\varphi(v)\mathrm{d}\widehat{N}(v)-\mu^{-1}\int_{[0,t]}\varphi(v)\mathrm{d}v}{\sqrt{k(t)}} \\
&= \frac{\varphi(t)(\widehat{N}(t)-\mu^{-1}t)}{\sqrt{k(t)}}+\frac{\int_{[0,t]}(\widehat{N}(v)-\mu^{-1}v)\mathrm{d}(-\varphi(v))}{\sqrt{k(t)}} \\
&= W_t(1)\frac{g(t)\varphi(t)}{\sqrt{k(t)}}+\frac{\int_{[0,t]}(\widehat{N}(v)-\mu^{-1}v)\mathrm{d}(-\varphi(v))}{\sqrt{k(t)}}.
\end{aligned}$$

By Lemma 5.3, $\lim_{t\to\infty}\frac{g(t)\varphi(t)}{\sqrt{k(t)}}=0$. Since, in view of (29), $W_t(1)\overset{d}{\to}Z(1)$, $t\to\infty$, we have

$$W_t(1)\frac{g(t)\varphi(t)}{\sqrt{k(t)}} \overset{P}{\to} 0, \quad t\to\infty.$$

Use now inequality (31) with $a=1$ and $x(t)=\sqrt{k(t)}$. Since, by Lemma 5.3, $\lim_{t\to\infty}\frac{\int_{[0,t]}g(v)\mathrm{d}(-\varphi(v))}{\sqrt{k(t)}}=0$ we conclude that

$$\frac{\int_{[0,t]}(\widehat{N}(v)-\mu^{-1}v)\mathrm{d}(-\varphi(v))}{\sqrt{k(t)}} \overset{P}{\to} 0, \quad t\to\infty.$$

The proof of the lemma is complete. □

Remark 3.5. Set

$$m(x):=\int_{[0,\,x]}\mathbb{P}\{|\log(1-W)|>y\}\mathrm{d}y, \quad x>0.$$

Lemma 5.4 in [10] proves that $|m(x)-k(x)|$ is a bounded function. This justifies the last sentence of Theorem 1.2. Also this implies that the normalization $\sqrt{\mu^{-1}k(t)}$ ($\sqrt{\mu^{-1}k(\log t)}$) used in Lemma 3.2, Lemma 3.3 and Lemma 3.4 can be safely replaced by $\sqrt{\mu^{-1}m(t)}$ ($\sqrt{\mu^{-1}m(\log t)}$).

Lemma 3.6. (I) *Assume that $\sigma^2=\infty$ and that*

$$\int_0^x y^2\mathbb{P}\{|\log W|\in\mathrm{d}y\} \sim \widetilde{\ell}(x), \quad x\to\infty,$$

for some $\widetilde{\ell}$ slowly varying at ∞. Let $c(x)$ be any positive function such that $\lim_{x\to\infty}\frac{x\widetilde{\ell}(c(x))}{c^2(x)}=1$. Assume further that

$$\mathbb{P}\{|\log(1-W)|>x\} \sim x^{-\beta}\ell(x), \quad x\to\infty,$$

for some $\beta\in[0,1/2)$ and some ℓ slowly varying at ∞. Then

$$\frac{C(t)-\mu^{-1}k(t)}{\mu^{-3/2}c(t)\varphi(t)} \overset{d}{\to} \int_{[0,1]}v^{-\beta}\mathrm{d}Z(v), \quad t\to\infty, \tag{34}$$

where $(Z(v))_{v\in[0,1]}$ is the Brownian motion.
(II) *Assume that*

$$\mathbb{P}\{|\log W|>x\} \sim x^{-\alpha}\widetilde{\ell}(x), \quad x\to\infty$$

for some $\alpha \in (1,2)$ *and some* $\widetilde{\ell}$ *slowly varying at* ∞. *Let* $c(x)$ *be any positive function such that* $\lim_{x\to\infty} \frac{x\widetilde{\ell}(c(x))}{c^\alpha(x)} = 1$. *Assume further that*

$$\mathbb{P}\{|\log(1-W) > x|\} \ \sim \ x^{-\beta}\ell(x), \quad x \to \infty,$$

for some $\beta \in [0, 1/\alpha)$ *and some* ℓ *slowly varying at* ∞. *Then*

$$\frac{C(t) - \mu^{-1}k(t)}{\mu^{-1-1/\alpha}c(t)\varphi(t)} \ \stackrel{d}{\to} \ \int_{[0,1]} v^{-\beta}\mathrm{d}Z(v), \quad t \to \infty, \tag{35}$$

where $(Z(v))_{v\in[0,1]}$ *is the* α*-stable Lévy process such that* $Z(1)$ *has characteristic function* (10).

Proof. The condition

$$\mathbb{P}\{|\log(1-W)| > x\} \ \sim \ x^{-\beta}\ell(x), \quad x \to \infty$$

is equivalent to the following

$$\mathbb{P}\{1 - W \le x\} \ \sim \ (\log(1/x))^{-\beta}\ell(\log(1/x)), \quad x \downarrow 0.$$

By Theorem 1.7.1' in [4], the latter is equivalent to

$$\varphi(t) \ \sim \ t^{-\beta}\ell(t), \quad t \to \infty. \tag{36}$$

Recalling the relation $g(t) \sim \mathrm{const}\, c(t)$, $t \to \infty$, we conclude that

$$\lim_{t\to\infty} g(t)\varphi(t) = \infty. \tag{37}$$

In view of Lemma 3.1 it suffices to prove that

$$C^*(t) := \frac{\widehat{C}(t) - \mu^{-1}k(t)}{g(t)\varphi(t)} \ \stackrel{d}{\to} \ \int_{[0,1]} v^{-\beta}\mathrm{d}Z(v), \quad t \to \infty.$$

Case $\beta = 0$. Recalling $W_t(1) \stackrel{d}{\to} Z(1)$, $t \to \infty$ and using the equality

$$C^*(t) = W_t(1) + \frac{\int_{[0,t]} (\widehat{N}(v) - \mu^{-1}v)\mathrm{d}(-\varphi(v))}{g(t)\varphi(t)}$$

we conclude that it remains to check that the second term in the right-hand side converges to zero in probability. According to (31) (with $a = 1$ and $x(t) = g(t)\varphi(t)$), it is enough to show that

$$\lim_{t\to\infty} \frac{\int_{[0,\,t]} g(v)\mathrm{d}\big(-\varphi(v)\big)}{g(t)\varphi(t)} = 0. \tag{38}$$

In view of Potter's bound (Theorem 1.5.6 in [4]), given $A > 0$ and $\delta \in (0, 1/\alpha - \beta)$ (here we take $\alpha = 2$ in the case (I) of the lemma) there exists $t_0 > 0$ such that

$$\frac{g(tu)}{g(t)} \le Au^{1/\alpha-\delta}, \tag{39}$$

whenever $t \ge t_0$, $tu \ge t_0$ and $u \le 1$. Since

$$\frac{\int_{[t_0,t]} g(v)\mathrm{d}(-\varphi(v))}{g(t)\varphi(t)} \ \stackrel{(39)}{\le} \ A\frac{\int_{[t_0,\,t]} v^{1/\alpha-\delta}\mathrm{d}(-\varphi(v))}{t^{1/\alpha-\delta}\varphi(t)} \ \to \ 0, \quad t \to \infty,$$

where the last relation is justified by Theorem 1.6.4 in [4], and

$$\lim_{t\to\infty}\frac{\int_{[0,t_0]}g(v)\mathrm{d}(-\varphi(v))}{g(t)\varphi(t)}=0,$$

which holds in view of (37), (38) follows.
CASE $\beta\neq 0$. We use the representation: for any fixed $\varepsilon>0$,

$$\begin{aligned}C^*(t) &= W_t(1)+\int_{[0,1]}W_t(v)\mu_t(\mathrm{d}v)\\ &= W_t(1)+\int_{[0,\,\varepsilon]}\ldots+\int_{[\varepsilon,\,1]}\ldots\\ &=: W_t(1)+I_1(\varepsilon,t)+I_2(\varepsilon,t),\end{aligned}$$

where the measure μ_t is defined by $\mu_t((v,1])=\varphi(vt)/\varphi(t)$, $v\in[0,1)$.

According to (36), as $t\to\infty$, μ_t converges weakly on $[\varepsilon,1]$ to a measure μ defined by $\mu((v,1]):=v^{-\beta}$. Together with (29) this entails the convergence

$$I_2(\varepsilon,t)\ \overset{d}{\to}\ \beta\int_{[\varepsilon,1]}Z(v)v^{-\beta-1}\mathrm{d}v,\quad t\to\infty$$

by Lemma 5.6. Further one can check that

$$W_t(1)+I_2(\varepsilon,t)\ \overset{d}{\to}\ Z(1)+\beta\int_{[\varepsilon,1]}Z(v)v^{-\beta-1}\mathrm{d}v,\quad t\to\infty.$$

According to Theorem 4.2 in [2], it remains to show that, for any $\gamma>0$,

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\mathbb{P}\{|I_1(\varepsilon,t)|>\gamma\}=0.$$

In view of inequality (31) (with $a=\varepsilon$ and $x(t)=g(t)\varphi(t)$) this will follow once we can prove that

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\frac{\int_{[0,\,\varepsilon t]}g(v)\mathrm{d}(-\varphi(v))}{g(t)\varphi(t)}=0. \tag{40}$$

Since $\lim\limits_{t\to\infty}\frac{\int_{[0,t_0]}g(v)\mathrm{d}(-\varphi(v))}{g(t)\varphi(t)}=0$ (use (37)) and

$$\frac{\int_{[t_0,t]}g(v)\mathrm{d}(-\varphi(v))}{g(t)\varphi(t)}\ \overset{(39)}{\le}\ A\frac{\int_{[t_0,\,t]}v^{1/\alpha-\delta}\mathrm{d}(-\varphi(v))}{t^{1/\alpha-\delta}\varphi(t)}\ \sim\ \frac{\beta}{1/\alpha-\beta-\delta}\varepsilon^{1/\alpha-\beta-\delta},$$

where the last relation is justified by Theorem 1.6.4 in [4], (40) follows. The proof of the lemma is finished. □

STEP 3. The purpose of this intermediate step is to combine results of the two previous steps into a single statement concerning convergence in distribution of $L(t)$, properly normalized and centered. In particular, we conclude that under the assumptions of the theorem

$$\frac{L(t)-b(t)}{a(t)}\ \overset{d}{\to}\ X,\quad t\to\infty, \tag{41}$$

for $b(t):=\mu^{-1}k(\log t)$, case-dependent normalizing function $a(t)$ and case-dependent random variable X. Now we identify the functions $a(t)$ and the laws of X for each case.

Cases (a), (b1) and (c1): $X \overset{d}{=} \mathcal{N}(0,1)$ and $a(t) = \sqrt{\mu^{-1}k(\log t)}$. This immediately follows from (33).
Case (b2): $X \overset{d}{=} \mathcal{N}(0,1)$ and $a(t) = \mu^{-3/2}c(\log t)\psi(t)$. By Lemma 3.3 and Lemma 3.6 (case $\beta = 0$),

$$\frac{L(t) - C(\log t)}{\sqrt{\mu^{-1}k(\log t)}} \overset{d}{\to} \mathcal{N}(0,1) \quad \text{and} \quad \frac{C(\log t) - \mu^{-1}k(\log t)}{\mu^{-3/2}c(\log t)\psi(t)} \overset{d}{\to} \mathcal{N}(0,1), \quad t \to \infty,$$

According to (36), $\varphi(t) \sim \ell(t)$, $t \to \infty$. Therefore, as $t \to \infty$, $k(t) \sim t\ell(t)$ (use Proposition 1.5.8 in [4]) and $c(t)\varphi(t) \sim t^{1/2}\ell^*(t)\ell(t)$ which, in view of the assumption $\lim_{t\to\infty} \ell(t)(\ell^*(t))^2 = \infty$, implies $\lim_{t\to\infty} \frac{\sqrt{k(\log t)}}{c(\log t)\psi(t)} = 0$. Hence,

$$\frac{L(t) - \mu^{-1}k(\log t)}{\mu^{-3/2}c(\log t)\psi(t)} \overset{d}{\to} \mathcal{N}(0,1), \quad t \to \infty.$$

Case (c2): $X \overset{d}{=} \int_{[0,1]} v^{-\beta}\mathrm{d}Z(v)$ and $a(t) = \mu^{-1-1/\alpha}c(\log t)\psi(t)$. By Lemma 3.3 and Lemma 3.6,

$$\frac{L(t) - C(\log t)}{\sqrt{\mu^{-1}k(\log t)}} \overset{d}{\to} \mathcal{N}(0,1) \quad \text{and} \quad \frac{C(\log t) - \mu^{-1}k(\log t)}{\mu^{-1-1/\alpha}c(\log t)\psi(t)} \overset{d}{\to} \int_{[0,1]} v^{-\beta}\mathrm{d}Z(v), \quad t \to \infty,$$

respectively. According to (36), $\varphi(t) \sim t^{-\beta}\ell(t)$, $t \to \infty$. Therefore, as $t \to \infty$, $k(t) \sim \mathrm{const}\, t^{1-\beta}\ell(t)$ by Proposition 1.5.8 in [4], and $c(t)\varphi(t) \sim t^{1/\alpha-\beta}\ell^*(t)\ell(t)$. While in the case $\beta \in [0, 2/\alpha - 1)$ the relation $\lim_{t\to\infty} \frac{\sqrt{k(\log t)}}{c(\log t)\psi(t)} = 0$ holds trivially, in the case $\beta = 2/\alpha - 1$ it is secured by the assumption $\lim_{t\to\infty} \ell(t)(\ell^*(t))^2 = \infty$. Hence,

$$\frac{L(t) - \mu^{-1}k(\log t)}{\mu^{-1-1/\alpha}c(\log t)\psi(t)} \overset{d}{\to} \int_{[0,1]} v^{-\beta}\mathrm{d}Z(v), \quad t \to \infty.$$

Step 4. Depoissonization. Since $L(\tau_n) = L_n$, where $(\tau_n)_{n\in\mathbb{N}}$ are arrival times of (π_t), it suffices to check that

$$\frac{L(\tau_n) - b(n)}{a(n)} \overset{d}{\to} X, \quad n \to \infty.$$

In the subsequent computations we will use arbitrary but fixed $x \in \mathbb{R}$. Given such an x we will choose $n_0 \in \mathbb{N}$ and $t_0 > 0$ such that $n \pm x\sqrt{n} \geq 0$ for $n \geq n_0$ and $t \pm x\sqrt{t} \geq 0$ for $t \geq t_0$. With this notation laid down all the inequalities or equalities that follow will be considered either for $t \geq t_0$ or $n \geq n_0$.

The functions $a(t)$ are slowly varying. While in the cases (b2) and (c2) this is trivial, in the remaining cases, as has already been mentioned, this follows from the equality $k(\log t) = \int_{[1,\,t]} \psi(y)y^{-1}\mathrm{d}y$ and Theorem 1.3.1 in [4]. The slow variation implies that the convergence $\lim_{t\to\infty} \frac{a(ty)}{a(t)} = 1$ takes place locally uniformly in y. In particular,

$$\lim_{t\to\infty} \frac{a(t \pm x\sqrt{t})}{a(t)} = 1. \tag{42}$$

The function $b(t)$ enjoys the following property

$$\lim_{t\to\infty} \big(b(t \pm x\sqrt{t}) - b(t)\big) = 0$$

which entails

$$\lim_{t\to\infty}\frac{b(t\pm x\sqrt{t})-b(t)}{a(t)}=0. \tag{43}$$

Indeed,

$$0\le b(t+x\sqrt{t})-b(t)=\mu^{-1}\int_{[t,t+x\sqrt{t}]}y^{-1}\psi(y)\mathrm{d}y\le \mu^{-1}\psi(t)\log(1+xt^{-1/2})\ \sim\ \mu^{-1}\psi(t)xt^{-1/2},$$

and the corresponding relation with 'minus' sign follows similarly.

Now (42) and (43) ensure that (41) is equivalent to

$$\frac{L(t\pm x\sqrt{t})-b(t)}{a(t)}\ \overset{d}{\to}\ X,\quad t\to\infty. \tag{44}$$

We will need the following observation

$$\frac{M(t+x\sqrt{t})-M(t-x\sqrt{t})}{a(t)}\ \overset{P}{\to}\ 0,\quad t\to\infty, \tag{45}$$

where the notation $M(t)=M_{\pi_t}$ has to be recalled. Actually, we can prove a stronger assertion

$$M(t+x\sqrt{t})-M(t-x\sqrt{t})\ \overset{P}{\to}\ 0,\quad t\to\infty,$$

as follows. Since $M(t)$ is nondecreasing it suffices to show that the expectation of the left-hand side converges to zero. To this end, we first prove the formula

$$\mathbb{E}M(t)=\mathbb{E}\sum_{k\ge 0}\big(1-\exp(-te^{-S_k})\big)=\mathbb{E}\int_{[0,\,\infty)}\big(1-\exp(-te^{-y})\big)\mathrm{d}N(y). \tag{46}$$

We use a variant of the random occupancy scheme with the random frequencies P_k's defined in the Introduction in which balls are thrown at the arrival times of the Poisson process (π_t). It is clear that $M(t)=0$ on the event $\{\pi_t=0\}$ and that

$$M(t)=\inf\{k\in\mathbb{N}:\pi_{k+1,\,t}+\pi_{k+2,\,t}+\ldots=0\}$$

on the event $\{\pi_t\ge 1\}$, where $\pi_{k,\,t}$ is the number of balls (out of π_t) falling in the kth box. Given $\big(P_k\big)$ $(\pi_{j,\,t})_{t\ge 0}$ is a Poisson process with intensity P_j, and, for different j, these Poisson processes are independent. With this at hand, it remains to write

$$\begin{aligned}\mathbb{E}\big(M(t)|(P_j)\big)\ &=\ \sum_{k\ge 0}\mathbb{P}\{M(t)>k|(P_j)\}=1-e^{-t}+\sum_{k\ge 1}\mathbb{P}\{\pi_{k+1,\,t}+\pi_{k+2,\,t}+\ldots\ge 1|(P_j)\}\\ &=\ 1-e^{-t}+\sum_{k\ge 1}\big(1-\exp(-t(1-P_1-\ldots-P_k))\big)\\ &=\ \sum_{k\ge 0}\big(1-\exp(-te^{-S_k})\big),\end{aligned}$$

and (46) follows on passing to the expectation. Using (46) we have

$$\begin{aligned}&\mathbb{E}\Big(M(t+x\sqrt{t})-M(t-x\sqrt{t})\Big)\\ =\ &\mathbb{E}\int_{[0,\,\infty)}\Big(\exp\big(-(t-x\sqrt{t})e^{-y}\big)-\exp\big(-(t+x\sqrt{t})e^{-y}\big)\Big)\mathrm{d}N(y)\\ \le\ &\frac{2x\sqrt{t}}{t-x\sqrt{t}}\mathbb{E}\int_{[0,\,\infty)}\exp\big(-(t-x\sqrt{t})e^{-y}\big)(t-x\sqrt{t})e^{-y}\mathrm{d}N(y)\\ \sim\ &2\mu^{-1}xt^{-1/2},\quad t\to\infty,\end{aligned}$$

where the last relation follows by the key renewal theorem (the function $t \mapsto \exp\big(t - e^t\big)$ is directly Riemann integrable on $\mathbb{R}$ since it is integrable on $\mathbb{R}$ and nonnegative, and $t \mapsto \exp(-e^t)$ is a nonincreasing function).

Further, setting $A_n(x) := \{|\tau_n - n| > x\sqrt{n}\}$ and recalling the notation $K(t) := K_{\pi_t}$, we have, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}\Big\{\frac{L(\tau_n) - L(n - x\sqrt{n})}{a(n)} > 2\varepsilon\Big\} &= \mathbb{P}\Big\{\frac{M(\tau_n) - K(\tau_n) - L(n - x\sqrt{n})}{a(n)} > 2\varepsilon\Big\} \\
&= \mathbb{P}\big\{\ldots 1_{A_n^c(x)} + \ldots 1_{A_n(x)} > 2\varepsilon\big\} \\
&\le \mathbb{P}\Big\{\frac{M(n + x\sqrt{n}) - K(n - x\sqrt{n}) - L(n - x\sqrt{n})}{a(n)} > \varepsilon\Big\} \\
&+ \mathbb{P}\big\{\ldots 1_{A_n(x)} > \varepsilon\big\} \\
&\le \mathbb{P}\Big\{\frac{M(n + x\sqrt{n}) - M(n - x\sqrt{n})}{a(n)} > \varepsilon\Big\} + \mathbb{P}\big(A_n(x)\big).
\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \mathbb{P}\Big\{\frac{L(\tau_n) - L(n - x\sqrt{n})}{a(n)} > 2\varepsilon\Big\} \le \mathbb{P}\big\{|\mathcal{N}(0,1)| > x\big\}, \tag{47}$$

by (45) and the central limit theorem. Since the law of X is continuous, we conclude that, for any $y \in \mathbb{R}$ and any $\varepsilon > 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}\Big\{\frac{L(\tau_n) - b(n)}{a(n)} > y\Big\} &\le \limsup_{n\to\infty} \mathbb{P}\Big\{\frac{L(\tau_n) - L(n - x\sqrt{n})}{a(n)} > 2\varepsilon\Big\} \\
&+ \lim_{n\to\infty} \mathbb{P}\Big\{\frac{L(n - x\sqrt{n}) - b(n)}{a(n)} > y - 2\varepsilon\Big\} \\
&\overset{(44),(47)}{\le} \mathbb{P}\big\{|\mathcal{N}(0,1)| > x\big\} + \mathbb{P}\big\{X > y - 2\varepsilon\big\}.
\end{aligned}$$

Letting now $x \to \infty$ and then $\varepsilon \downarrow 0$ gives

$$\limsup_{n\to\infty} \mathbb{P}\Big\{\frac{L(\tau_n) - b(n)}{a(n)} > y\Big\} \le \mathbb{P}\big\{X > y\big\}.$$

Arguing similarly we infer

$$\limsup_{n\to\infty} \mathbb{P}\Big\{\frac{L(n + x\sqrt{n}) - L(\tau_n)}{a(n)} > 2\varepsilon\Big\} \le \mathbb{P}\big\{|\mathcal{N}(0,1)| > x\big\} \tag{48}$$

and then

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{P}\Big\{\frac{L(\tau_n) - b(n)}{a(n)} > y\Big\} &\ge \lim_{n\to\infty} \mathbb{P}\Big\{\frac{L(n + x\sqrt{n}) - b(n)}{a(n)} > y + 2\varepsilon\Big\} \\
&- \limsup_{n\to\infty} \mathbb{P}\Big\{\frac{L(n + x\sqrt{n}) - L(\tau_n)}{a(n)} > 2\varepsilon\Big\} \\
&\overset{(44),(48)}{\ge} \mathbb{P}\big\{X > y + 2\varepsilon\big\} - \mathbb{P}\big\{|\mathcal{N}(0,1)| > x\big\}.
\end{aligned}$$

Letting $x \to \infty$ and then $\varepsilon \downarrow 0$ we arrive at

$$\liminf_{n\to\infty} \mathbb{P}\Big\{\frac{L(\tau_n) - b(n)}{a(n)} > y\Big\} \ge \mathbb{P}\big\{X > y\big\}.$$

The proof of Theorem 1.2 is complete.

Remark 3.7. Here we discuss what is known in cases (b3) and (c3) introduced in Remark 1.4.
Case (b3): By Lemma 3.3 and Lemma 3.6,

$$\frac{L(t)-C(\log t)}{\sqrt{\mu^{-1}k(\log t)}} \overset{d}{\to} X_1 \quad \text{and} \quad \frac{C(\log t)-\mu^{-1}k(\log t)}{\mu^{-3/2}c(\log t)\psi(t)} \overset{d}{\to} X_2, \quad t\to\infty, \tag{49}$$

respectively, where X_1 and X_2 are random variables with the standard normal distribution. According to (36), $\varphi(t) \sim d(\ell^*(t))^{-2}$, $t\to\infty$. Therefore, as $t\to\infty$, $k(t)\sim dt(\ell^*(t))^{-2}$ (use Proposition 1.5.8 in [4]) and $c(t)\varphi(t)\sim dt^{1/2}(\ell^*(t))^{-1}$. Consequently, (49) is equivalent to

$$\frac{\ell^*(\log t)}{\log^{1/2} t}\big(L(t)-C(\log t)\big) \overset{d}{\to} (d/\mu)^{1/2}X_1 \quad \text{and} \quad \frac{\ell^*(\log t)}{\log^{1/2} t}\big(C(\log t)-\mu^{-1}k(\log t)\big) \overset{d}{\to} d\mu^{-3/2}X_2, \quad t\to\infty.$$

However, we do not know whether the joint convergence of these ratios takes place, nor do we know how dependent the random variables X_1 and X_2 are. The same remark concerns formula (51) given below.
Case (c3): By Lemma 3.3 and Lemma 3.6,

$$\frac{L(t)-C(\log t)}{\sqrt{\mu^{-1}k(\log t)}} \overset{d}{\to} X_1 \quad \text{and} \quad \frac{C(\log t)-\mu^{-1}k(\log t)}{\mu^{-1-1/\alpha}c(\log t)\psi(t)} \overset{d}{\to} X_2, \quad t\to\infty, \tag{50}$$

respectively, where $X_1 \overset{d}{=} \mathcal{N}(0,1)$ and $X_2 \overset{d}{=} \int_{[0,1]} v^{1-2/\alpha}\mathrm{d}Z(v)$. According to (36), $\varphi(t) \sim dt^{1-2/\alpha}(\ell^*(t))^{-2}$, $t\to\infty$. Therefore, as $t\to\infty$, $k(t)\sim d(2-2/\alpha)^{-1}t^{2-2/\alpha}(\ell^*(t))^{-2}$ by Proposition 1.5.8 in [4], and $c(t)\varphi(t)\sim dt^{1-1/\alpha}(\ell^*(t))^{-1}$. Consequently, (50) is equivalent to

$$\frac{\ell^*(t)}{t^{1-1/\alpha}}\big(L(t)-C(\log t)\big) \overset{d}{\to} (2\mu(1-1/\alpha)/d)^{-1/2}X_1, \quad \frac{\ell^*(t)}{t^{1-1/\alpha}}\big(C(\log t)-\mu^{-1}k(\log t)\big) \overset{d}{\to} d\mu^{-1-1/\alpha}X_2. \tag{51}$$

It seems that in order to settle the weak convergence issue in these cases one has to investigate the weak convergence of $L^*(t)=\sum_{k\ge1}\exp(-te^{-S_{k-1}}(1-W_k))1_{\{S_{k-1}\le \log t\}}$ directly, i.e. without using the decomposition $L^*(t)=\big(L^*(t)-C(\log t)\big)+C(\log t)$.

4 Answering a question asked in [19]

Let $\big(\xi_k,\eta_k\big)_{k\in\mathbb{N}}$ be independent copies of a random vector (ξ,η) with $\xi>0$ and $\eta\ge0$ a.s. An arbitrary dependence between ξ and η is allowed. In what follows we also assume that $\mathtt{m}:=\mathbb{E}\xi<\infty$, $\mathbb{E}\eta=\infty$, and that the law of ξ is non-lattice. Set

$$V(t):=\sum_{k\ge1}1_{\{\widetilde{S}_{k-1}\le t<\widetilde{S}_{k-1}+\eta_k\}}, \quad t\ge0,$$

where

$$\widetilde{S}_0:=0, \quad \widetilde{S}_k:=\xi_1+\ldots+\xi_k, \quad k\in\mathbb{N}.$$

Assuming that ξ and η are independent and that $\bar{G}(x):=\mathbb{P}\{\eta>x\}$ is regularly varying at ∞ with index $-\beta$, $\beta\in[0,1)$, Proposition 3.2 in [19] proves[3] that

$$\frac{V(t)-\sum_{k\ge1}\bar{G}(t-\widetilde{S}_{k-1})1_{\{\widetilde{S}_{k-1}\le t\}}}{\sqrt{\mathtt{m}^{-1}\int_{[0,\,t]}\bar{G}(y)\mathrm{d}y}} \overset{d}{\to} \mathcal{N}(0,1), \quad t\to\infty. \tag{52}$$

[3] Actually the cited result treats the finite-dimensional convergence.

In Problem 1 of Section 5.2 in [19] the authors ask "when can the random centering be replaced by a non-random centering?" Relying on the results developed in Section 3 we can answer this question in an extended setting where ξ and η are not necessarily independent. In particular, the replacement is possible, i.e.,

$$\frac{V(t)-\mathtt{m}^{-1}\int_{[0,\,t]}\bar{G}(y)\mathrm{d}y}{\sqrt{\mathtt{m}^{-1}\int_{[0,\,t]}\bar{G}(y)\mathrm{d}y}} \stackrel{d}{\to} \mathcal{N}(0,1), \quad t\to\infty,$$

if either of the following three conditions holds:

- $\mathbb{E}\xi^2<\infty$
- $\mathbb{E}\xi^2=\infty$, $\int_{[0,\,x]}y^2\mathbb{P}\{\xi\in\mathrm{d}y\}\sim\widetilde{\ell}(x)$, $x\to\infty$, where $\widetilde{\ell}$ is slowly varying at ∞, and $\lim_{x\to\infty}\bar{G}(x)c^2(x)x^{-1}=0$, where $c(x)$ is any positive function which satisfies $\lim_{x\to\infty}\frac{x\widetilde{\ell}(c(x))}{c^2(x)}=1$
- $\mathbb{P}\{\xi>x\}\sim x^{-\alpha}\widetilde{\ell}(x)$, $x\to\infty$ for some $\alpha\in(1,2)$ and some $\widetilde{\ell}$ slowly varying at ∞ and $\lim_{x\to\infty}\bar{G}(x)c^2(x)x^{-1}=0$, where $c(x)$ is any positive function which satisfies $\lim_{x\to\infty}\frac{x\widetilde{\ell}(c(x))}{c^\alpha(x)}=1$

The replacement is not possible if either of the following two conditions holds:

- $\mathbb{E}\xi^2=\infty$, $\int_{[0,\,x]}y^2\mathbb{P}\{\xi\in\mathrm{d}y\}\sim\widetilde{\ell}(x)$, $x\to\infty$, where $\widetilde{\ell}$ is slowly varying at ∞; $\bar{G}(x)\sim\ell(x)$, $x\to\infty$, where ℓ is slowly varying at ∞, and $\lim_{x\to\infty}\bar{G}(x)c^2(x)x^{-1}=\infty$, where $c(x)$ is any positive function which satisfies $\lim_{x\to\infty}\frac{x\widetilde{\ell}(c(x))}{c^2(x)}=1$
- $\mathbb{P}\{\xi>x\}\sim x^{-\alpha}\widetilde{\ell}(x)$, $x\to\infty$, for some $\alpha\in(1,2)$ and some $\widetilde{\ell}$ slowly varying at ∞; $\bar{G}(x)\sim x^{-\beta}\ell(x)$, $x\to\infty$, for some $\beta\in[0,2/\alpha-1]$ and some ℓ slowly varying at ∞; $\lim_{x\to\infty}\bar{G}(x)c^2(x)x^{-1}=\infty$ if $\beta=2/\alpha-1$, where $c(x)$ is any positive function which satisfies $\lim_{x\to\infty}\frac{x\widetilde{\ell}(c(x))}{c^\alpha(x)}=1$

In these cases

$$\frac{V(t)-\mathtt{m}^{-1}\int_{[0,\,t]}\bar{G}(y)\mathrm{d}y}{\mathtt{m}^{-1-1/\alpha}c(t)\bar{G}(t)} \stackrel{d}{\to} X, \quad t\to\infty,$$

where in the first case $\alpha=2$ and $X\stackrel{d}{=}\mathcal{N}(0,1)$, and in the second case $X\stackrel{d}{=}\int_{[0,1]}v^{-\beta}\mathrm{d}Z(v)$, where $(Z(v))_{v\in[0,1]}$ is an α-stable Lévy process with characteristic function (10).

To justify these statements we first note that mimicking the proof of Lemma 3.3 we can check that relation (52) holds under the standing assumptions of this section. Let E be a random variable with the standard exponential distribution which is independent of everything else. We claim that

$$-\int_{[1,\,\infty)}\widetilde{N}(\log x)e^{-x}\mathrm{d}x \stackrel{d}{\le} R(t):=\sum_{k\ge 0}\big(\bar{G}(t-\widetilde{S}_k)-\widehat{\varphi}(t-\widetilde{S}_k)\big)1_{\{\widetilde{S}_k\le t\}} \stackrel{d}{\le} \int_{[0,\,1]}\widetilde{N}(|\log x|)e^{-x}\mathrm{d}x, \tag{53}$$

where

$$\widetilde{N}(t):=\inf\{k\in\mathbb{N}_0:\widetilde{S}_k>t\} \quad\text{and}\quad \widehat{\varphi}(t):=\mathbb{E}\exp(-e^{t-\eta}),\quad t\ge 0.$$

Using the subadditivity of $t \to t^+$, $t \in \mathbb{R}$ and the distributional subadditivity of $\widetilde{N}(t)$ (see (19)) we obtain

$$\begin{aligned}\int_{[0,\,t]} \big(1_{\{\eta>t-y\}} - 1_{\{\log E+\eta>t-y\}}\big)\mathrm{d}\widetilde{N}(y) &= \widetilde{N}\big((t-\eta-\log E)^+\big) - \widetilde{N}\big((t-\eta)^+\big)\\ &\leq \widetilde{N}\big((t-\eta)^+ + (\log E)^-\big) - \widetilde{N}\big((t-\eta)^+\big)\\ &\overset{d}{\leq} \widetilde{N}\big((\log E)^-\big).\end{aligned}$$

Hence

$$\begin{aligned}R(t) &= \mathbb{E}_{\eta,E}\int_{[0,\,t]} \big(1_{\{\eta>t-y\}} - 1_{\{\log E+\eta>t-y\}}\big)\mathrm{d}\widetilde{N}(y)\\ &\overset{d}{\leq} \mathbb{E}_{\eta,E}\widetilde{N}\big((\log E)^-\big) = \int_{[0,\,1]} \widetilde{N}(|\log x|)e^{-x}\mathrm{d}x.\end{aligned}$$

The lower bound in (53) can be proved similarly.

With (53) at hand, we conclude that Lemma 3.4 and Lemma 3.6 are still valid if $\varphi(t)$ is replaced by $\bar{G}(t)$ and $C(t)$ is replaced by $\sum_{k\geq 1}\bar{G}(t-\widetilde{S}_{k-1})1_{\{\widetilde{S}_{k-1}\leq t\}}$. It remains to combine these generalizations of Lemma 3.4 and Lemma 3.6 and our extended version of (52).

5 Appendix

Lemma 5.1 which is our main technical tool for proving Theorem 2.1 is a rather particular case of a Toeplitz- Schur theorem (see Lemma 8.1 in [12]). On the other hand, this result follows immediately by an application of the Lebesgue bounded convergence theorem.

Lemma 5.1. *Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\lim_{n\to\infty} s_n = s \in (0,\infty)$ and $(c_{n,\,m})_{n\in\mathbb{N},m\in\mathbb{N}}$ an array of nonnegative numbers which satisfy (A) $\lim_{n\to\infty} c_{n,\,m} = 0$, for each $m \in \mathbb{N}$, and (B) $\sum_{m=1}^n c_{n,\,m} = 1$. Then $\lim_{n\to\infty}\sum_{m=1}^n c_{n,\,m}s_m = s$.*

Lemma 5.2. *Let ξ and η be positive random variables. The relation*

$$\lim_{x\downarrow 0}\frac{\mathbb{P}\{\xi\leq x\}}{\mathbb{P}\{\eta\leq x\}} = c \in [0,\infty]$$

entails

$$\lim_{y\to\infty}\frac{\mathbb{E}e^{-y\xi}}{\mathbb{E}e^{-y\eta}} = c.$$

Proof. By symmetry, it suffices to consider the case $c \in [0,\infty)$. For any $\varepsilon > 0$ there exists $x_0 > 0$ such that $\mathbb{P}\{\xi\leq x\}/\mathbb{P}\{\eta\leq x\} \leq c+\varepsilon$ for all $x \in (0, x_0]$. With this x_0 we have

$$\begin{aligned}\frac{\mathbb{E}e^{-y\xi}}{\mathbb{E}e^{-y\eta}} &\leq \frac{\int_0^{x_0} e^{-yx}\mathbb{P}\{\xi\leq x\}\mathrm{d}x + \int_{x_0}^\infty e^{-yx}\mathbb{P}\{\xi\leq x\}\mathrm{d}x}{\int_0^{x_0} e^{-yx}\mathbb{P}\{\eta\leq x\}\mathrm{d}x}\\ &\leq \frac{(c+\varepsilon)\int_0^{x_0} e^{-yx}\mathbb{P}\{\eta\leq x\}\mathrm{d}x + y^{-1}e^{-yx_0}}{\int_0^{x_0} e^{-yx}\mathbb{P}\{\eta\leq x\}\mathrm{d}x}\\ &\leq (c+\varepsilon) + \frac{y^{-1}e^{-yx_0}}{\int_{x_0/2}^{x_0} e^{-yx}\mathbb{P}\{\eta\leq x\}\mathrm{d}x}\\ &\leq (c+\varepsilon) + \frac{1}{\mathbb{P}\{\eta\leq x_0/2\}(e^{yx_0/2}-1)}.\end{aligned}$$

Sending $y \to \infty$ and then $\varepsilon \to 0$ proves

$$\limsup_{y\to\infty} \frac{\mathbb{E}e^{-y\xi}}{\mathbb{E}e^{-y\eta}} \le c.$$

The lower limit (when $c > 0$) can be treated similarly. $\square$

Before stating the next result we recall notation: $\varphi(t) = \mathbb{E}\exp(-e^t(1-W))$, $k(t) = \int_{[0,t]} \varphi(y)\mathrm{d}y$. The functions $g(t)$ were defined in the paragraph preceding Lemma 3.4.

Lemma 5.3. *Assume that $\nu = \infty$ and that either conditions* (4) *and* (5) *or* (7) *and* (8) *hold, or $\sigma^2 < \infty$. Then*

$$\lim_{t\to\infty} \frac{g(t)\varphi(t)}{\sqrt{k(t)}} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\int_{[0,t]} g(y)\mathrm{d}(-\varphi(y))}{\sqrt{k(t)}} = 0. \tag{54}$$

Proof. Case $\sigma^2 < \infty$. The first relation in (54) is immediate:

$$\frac{g^2(t)\varphi^2(t)}{k(t)} = \mathrm{const}\,\frac{t\varphi^2(t)}{k(t)} \le \mathrm{const}\,\varphi(t) \ \to \ 0, \quad t \to \infty.$$

Condition $\nu = \infty$ is equivalent to $\lim_{t\to\infty} k(t) = \infty$. Therefore if the integral $\int_{[0,\infty)} y^{1/2}\mathrm{d}(-\varphi(y))$ converges the second relation in (54) holds trivially. Assume that $\lim_{t\to\infty} \int_{[0,t]} y^{1/2}\mathrm{d}(-\varphi(y)) = \infty$. Integrating by parts, we have

$$\frac{1}{\sqrt{k(t)}} \int_{[0,\,t]} y^{1/2}\mathrm{d}(-\varphi(y)) \ \sim \ \frac{1}{2\sqrt{k(t)}} \int_{[0,\,t]} \varphi(y)y^{-1/2}\mathrm{d}y, \quad t \to \infty.$$

By l'Hôpital rule,

$$\frac{1}{\sqrt{k(t)}} \int_{[1,\,t]} \varphi(y)y^{-1/2}\mathrm{d}y \ \sim \ 2\sqrt{k(t)/t} \ \to \ 0, \quad t \to \infty,$$

which proves the second relation in (54).
Case when conditions (4) and (5) hold. Let η be a random variable with distribution such that

$$\mathbb{P}\{\eta \le x\} \ \sim \ \frac{1}{(\ell^*(-\log x))^2}, \quad x \downarrow 0.$$

Then (5) is equivalent to

$$\lim_{x\downarrow 0} \frac{\mathbb{P}\{1-W \le x\}}{\mathbb{P}\{\eta \le x\}} = 0.$$

By Lemma 5.2,

$$\lim_{t\to\infty} \frac{\psi(t)}{\mathbb{E}e^{-t\eta}} = 0.$$

Since, by Theorem 1.7.1' in [4], $\mathbb{E}e^{-t\eta} \sim (\ell^*(\log t))^{-2}$, $t \to \infty$, we conclude that

$$\lim_{t\to\infty} \varphi(t)(\ell^*(t))^2 = 0.$$

Hence

$$\frac{g^2(t)\varphi^2(t)}{k(t)} \sim \mathrm{const}\,\frac{t(\ell^*(t))^2\varphi^2(t)}{k(t)} \le \mathrm{const}\,\varphi(t)(\ell^*(t))^2 \ \to \ 0, \quad t \to \infty.$$

If the integral $\int_{[0,\infty)} g(y)\mathrm{d}(-\varphi(y))$ converges the second relation in (54) holds trivially. Assume that $\lim_{t\to\infty} \int_{[0,t]} g(y)\mathrm{d}(-\varphi(y)) = \infty$. According to Theorem 1.8.3 in [4], we can assume, without loss of generality, that g is differentiable. Then, $g'(t) \sim \mathrm{const}\, t^{-1/2}\ell^*(t)$, $t \to \infty$. Integrating by parts, we have

$$\frac{1}{\sqrt{k(t)}} \int_{[0,t]} g(y)\mathrm{d}(-\varphi(y)) \ \sim\ \frac{1}{\sqrt{k(t)}} \int_{[1,t]} \varphi(y)g'(y)\mathrm{d}y, \quad t \to \infty.$$

By l'Hôpital rule,

$$\frac{1}{\sqrt{k(t)}} \int_{[1,\,t]} \varphi(y)g'(y)\mathrm{d}y \ \sim\ 2g'(t)\sqrt{k(t)} \ \sim\ \mathrm{const}\,\frac{g(t)}{t}\sqrt{k(t)}, \quad t \to \infty. \tag{55}$$

If $\varphi(t) \sim (\ell^*(t))^{-2}$, $t \to \infty$ then, by Proposition 1.5.8 in [4], $\lim_{t\to\infty} \frac{g(t)\sqrt{k(t)}}{t} = 1$. Therefore, the right-hand side of (55) goes to zero, as $t \to \infty$, if condition (5) holds.

The case when conditions (7) and (8) hold can be treated similarly, and we omit details. $\square$

Let $(S_k^*)_{k\in\mathbb{N}_0}$ be a zero-delayed random walk with positive steps. Set

$$N^*(x) := \inf\{k \in \mathbb{N}_0 : S_k^* > x\}, \quad x \geq 0.$$

Lemma 5.4 is used in Section 3 for investigating the asymptotics of moments.

Lemma 5.4. *Suppose $\mathbb{E}S_1^* < \infty$, and the law of S_1^* is non-lattice.*
(a) Let $r : [0,\infty) \to [0,\infty)$ be a nonincreasing function such that

$$\lim_{t\to\infty} \int_{[0,\,t]} r(y)\mathrm{d}y = \infty.$$

Then

$$\mathbb{E} \int_{[0,\,t]} r(t-z)\mathrm{d}N^*(z) \ \sim\ (\mathbb{E}S_1^*)^{-1} \int_{[0,\,t]} r(z)\mathrm{d}z, \quad t \to \infty.$$

(b) Let $r_1, r_2 : [0,\infty) \to [0,\infty)$ be nondecreasing functions such that $r_1(t) \geq r_2(t)$, $t \geq 0$, and

$$\lim_{t\to\infty} \int_{[0,\,t]} \big(r_1(y) - r_2(y)\big)\mathrm{d}y = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{r_1(t) + r_2(t)}{\int_{[0,\,t]} \big(r_1(y) - r_2(y)\big)\mathrm{d}y} = 0. \tag{56}$$

Then

$$\mathbb{E} \int_{[0,\,t]} \big(r_1(t-z) - r_2(t-z)\big)\mathrm{d}N^*(z) \ \sim\ (\mathbb{E}S_1^*)^{-1} \int_{[0,\,t]} \big(r_1(z) - r_2(z)\big)\mathrm{d}z, \quad t \to \infty.$$

Remark 5.5. The conclusion of Lemma 5.4(b) is in force whenever r_1 is a nondecreasing function of *subexponential growth* satisfying $\int_{[0,\infty)} r_1(y)\mathrm{d}y = \infty$ and $r_2 \equiv 0$.

Let us further note that the second condition in (56) cannot be omitted. Indeed, assuming that $r_1(t) = e^t$ and $r_2 \equiv 0$ we infer $\mathbb{E}\int_{[0,\,t]} r_1(t-y)\mathrm{d}N^*(y) \sim (1 - \mathbb{E}e^{-S_1^*})^{-1}e^t$, whereas $(\mathbb{E}S_1^*)^{-1} \int_{[0,\,t]} r_1(y)\mathrm{d}y \sim (\mathbb{E}S_1^*)^{-1}e^t$.

Part (a) of Lemma 5.4 is a fragment of Theorem 4 in [21]. The proof of part (b) requires only minor modifications and is thus omitted.

Lemma 5.6. *Let $0 \le a < b < \infty$. Assume that $X_t(\cdot) \Rightarrow X(\cdot)$, as $t \to \infty$, in $D[a,b]$ in the M_1 topology. Assume also that, as $t \to \infty$, μ_t converges weakly to μ on $[a,b]$, where (μ_t) is a family of Radon measures, and the limiting measure μ is absolutely continuous with respect to the Lebesgue measure. Then*

$$\int_{[a,b]} X_t(\cdot)\mu_t(\mathrm{d}y) \overset{d}{\to} \int_{[a,b]} X(\cdot)\mu(\mathrm{d}y), \quad t \to \infty.$$

Proof. It suffices to prove that

$$\lim_{t\to\infty} \int_{[a,b]} h_t(y)\mu_t(\mathrm{d}y) = \int_{[a,b]} h(y)\mu(\mathrm{d}y), \tag{57}$$

whenever $\lim_{t\to\infty} h_t(y) = h(y)$ in $D[a,b]$ in the M_1 topology, for the desired result then follows by the continuous mapping theorem.
Since $h \in D[a,b]$ the set D_h of its discontinuities is at most countable. By Lemma 12.5.1 in [23], convergence in the M_1 topology implies local uniform convergence at all continuity points of the limit. Hence $E := \{x : \text{there exists } x_t \text{ such that } \lim_{t\to\infty} x_t = x, \text{but } \lim_{t\to\infty} h_t(x_t) \neq h(x)\} \subseteq D_h$, and we conclude that $\mu(E) = 0$. Now (57) follows from Lemma 2.1 in [5]. $\square$

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